The random variables $X_1, \ldots, X_n$ are said to be *independent random variables* if

$$F(\mathbf{x}) = \prod_{i=1}^{n} F_i(x_i),$$

where $F_i(x_i) = P(X_i \le x_i)$ (which is referred to as the *marginal distribution* of X_i). If, in addition, $F_i(x_i) = F(x_i)$ for all i, then the random variables are said to be *independent and identically distributed*—or *i.i.d.* for short.

Expectations and Moment-Generating Functions

The *expected value*—or *mean*—of a random variable X is defined by the integral

$$E[X] = \int_{\Re} x dF(x),$$

where the right-hand side above is equal to $\int_{\Re} x f(x) dx$ if X is continuous and $\sum_x x P\{X = x\}$ when X is discrete. For a general function $g : \Re \to \Re$, the expected value of $g(X)$ is defined by

$$E[g(X)] = \int_{\Re} g(x) dF(x).$$

The *variance* of X is defined as

$$Var(X) = E[(X - E[X])^2].$$

If X and Y are two random variables, the *covariance* is defined by

$$Cov(X, Y) = E[(X - E[X])(Y - E[Y])].$$

The *moment-generating function* of X is defined as

$$\psi_X(t) = E[e^{tX}].$$

The n^{th} *moment* of X is defined as $E[X^n]$. If the moment-generating function exists then one can determine the n^{th} moment of X using the fact that

$$E[X^n] = \frac{\partial^n}{\partial t} \psi_X(t)|_{t=0},$$

where $\frac{\partial^n}{\partial t}$ denotes the n^{th} derivative with respect to t.

The moment-generating function is also useful for analyzing sums of random variables. Indeed, if X and Y are two independent random variables with moment-generating functions $\psi_X(t)$ and $\psi_Y(t)$, respectively, and $Z = X + Y$, then

$$\psi_Z(t) = \psi_X(t)\,\psi_Y(t).$$

That is, the moment-generating function of a sum of independent random variables is simply the product of their individual moment-generating functions.